UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1

May/June 2006

3 hours

Additional Materials: Answer Booklet/Paper
Graph paper
List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of **4** printed pages.

UNIVERSITY of CAMBRIDGE International Examinations

1 Express

$$u_n = \frac{1}{4n^2 - 1}$$

in partial fractions, and hence find $\sum_{n=1}^{N} u_n$ in terms of N. [4]

Deduce that the infinite series $u_1 + u_2 + u_3 + \dots$ is convergent and state the sum to infinity. [2]

2 Draw a diagram to illustrate the region R which is bounded by the curve whose polar equation is $r = \cos 2\theta$ and the lines $\theta = 0$ and $\theta = \frac{1}{6}\pi$. [2]

Determine the exact area of R. [4]

3 Prove by induction, or otherwise, that

$$23^{2n} + 31^{2n} + 46$$

is divisible by 48, for all integers $n \geqslant 0$. [6]

4 The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -3 & 4 & 5 \\ 5 & -6 & 10 & 14 \\ 4 & -5 & 8 & 11 \end{pmatrix}.$$

Show that the dimension of the range space of T is 2. [3]

Let $\mathbf{M}$ be a given 4×4 matrix and let S be the vector space consisting of vectors of the form $\mathbf{MAx}$, where $\mathbf{x} \in \mathbb{R}^4$. Show that if $\mathbf{M}$ is non-singular then the dimension of S is 2. [4]

5 The curve C has equation

$$y = 2x + \frac{3(x-1)}{x+1}.$$

(i) Write down the equations of the asymptotes of C. [2]

(ii) Find the set of values of x for which C is above its oblique asymptote and the set of values of x for which C is below its oblique asymptote. [3]

(iii) Draw a sketch of C, stating the coordinates of the points of intersection of C with the coordinate axes. [4]

6 **(a)** The equation of a curve is

$$y = \frac{2\sqrt{3}}{3}x^{\frac{3}{2}}.$$

Find the length of the arc of the curve from the origin to the point where $x = 1$. [4]

(b) The variables x and y are such that

$$y^3 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 = x^4 + 6.$$

Given that $y = -1$ when $x = 1$, find the value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x = 1$. [5]

7 Given that

$$I_n = \int_0^{\frac{1}{2}\pi} \sin^n x \,\mathrm{d}x,$$

where $n \geqslant 0$, prove that

$$I_{n+2} = \left(\frac{n+1}{n+2}\right)I_n.$$ [4]

The region bounded by the x-axis and the arc of the curve $y = \sin^4 x$ from $x = 0$ to $x = \pi$ is denoted by R. Determine the y-coordinate of the centroid of R. [5]

8 Obtain the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 6\frac{\mathrm{d}y}{\mathrm{d}t} + 25y = 80\mathrm{e}^{-3t}.$$ [5]

Given that $y = 8$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = -8$ when $t = 0$, show that $0 \leqslant y\mathrm{e}^{3t} \leqslant 10$ for all t. [5]

9 Given that $z = \mathrm{e}^{\mathrm{i}\theta}$ and n is a positive integer, show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad \text{and} \qquad z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta.$$ [2]

Hence express $\cos^7\theta$ in the form

$$p\cos 7\theta + q\cos 5\theta + r\cos 3\theta + s\cos\theta,$$

where the constants p, q, r, s are to be determined. [4]

Find the mean value of $\cos^7 2\theta$ with respect to θ over the interval $0 \leqslant \theta \leqslant \frac{1}{4}\pi$, leaving your answer in terms of π. [5]

10 The equation of the plane Π is

$$2x + 3y + 4z = 48.$$

Obtain a vector equation of Π in the form

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are of the form $p\mathbf{i}$, $q\mathbf{i} + r\mathbf{j}$ and $s\mathbf{i} + t\mathbf{k}$ respectively, and where p, q, r, s, t are integers. [6]

The line l has vector equation $\mathbf{r} = 29\mathbf{i} - 2\mathbf{j} - \mathbf{k} + \theta(5\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$. Show that l lies in Π. [3]

Find, in the form $ax + by + cz = d$, the equation of the plane which contains l and is perpendicular to Π. [4]

11 Answer only **one** of the following two alternatives.

EITHER

Obtain the sum of the squares of the roots of the equation

$$x^4 + 3x^3 + 5x^2 + 12x + 4 = 0.$$ [2]

Deduce that this equation does not have more than 2 real roots. [3]

Show that, in fact, the equation has exactly 2 real roots in the interval $-3 < x < 0$. [5]

Denoting these roots by α and β, and the other 2 roots by γ and δ, show that $|\gamma| = |\delta| = \dfrac{2}{\sqrt{(\alpha\beta)}}$. [4]

OR

The square matrix $\mathbf{A}$ has λ as an eigenvalue with corresponding eigenvector $\mathbf{x}$. The non-singular matrix $\mathbf{M}$ is of the same order as $\mathbf{A}$. Show that $\mathbf{Mx}$ is an eigenvector of the matrix $\mathbf{B}$, where $\mathbf{B} = \mathbf{MAM}^{-1}$, and that λ is the corresponding eigenvalue. [3]

It is now given that

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ a & -3 & 0 \\ b & c & -5 \end{pmatrix} \quad \text{and} \quad \mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Write down the eigenvalues of $\mathbf{A}$ and obtain corresponding eigenvectors in terms of a, b, c. [4]

(ii) Find the eigenvalues and corresponding eigenvectors of $\mathbf{B}$. [4]

(iii) Hence find a matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{B}^n = \mathbf{QDQ}^{-1}$. [3]

[You are not required to find $\mathbf{Q}^{-1}$.]